# Systematic Derivation of Anisotropic PML Absorbing Media in Cylindrical and Spherical Coordinates

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***Abstract*— A simple and systematic derivation of anisotropic perfectly matched layers (PML's) in cylindrical and spherical coordinates is presented. The derivation is based on the analytic continuation of Maxwell's Equations to complex space. Through field transformations, results for Cartesian anisotropic PML media are recovered and, more importantly, a generalization of the anisotropic PML to cylindrical and spherical systems is obtained, providing further clarification on the PML concept. As expected, these new PML media are cylindrically and spherically layered, respectively.**

***Index Terms*—Absorbing boundary conditions, anisotropic media, perfectly matched layer.**

## I. Introduction

THE perfectly matched layer (PML) absorbing boundary condition, first derived for Cartesian coordinates and planar interfaces [1], was recently extended to cylindrical [2]–[5] and spherical coordinates [2], [3]. In [2] and [3] this was achieved through an analytic continuation of the frequency-domain Maxwell's equations to complex space. As a result, the resultant fields *inside* the PML are *not* Maxwellian and the question naturally arises if it is possible to derive a Maxwellian anisotropic PML medium on cylindrical and spherical coordinates, as done for the Cartesian case [6], [7]. An anisotropic-medium formulation has the advantage of providing a physical basis for possible engineered artificial materials [8] and an easier interfacing with methods other than the finite-difference time-domain (FDTD), e.g., the finite-element method (FEM) [9], [10].

Here, a systematic analytical approach to derive the constitutive tensors for anisotropic PML formulations on Cartesian, cylindrical, and spherical coordinates from the complex space Maxwell's Equations is developed. The relation between the anisotropic PML fields and the complex space PML fields on each of these systems is elucidated by presenting the pertinent mapping equations.

From the constitutive tensors obtained, it explains why a previously proposed set of tensors for cylindrical and spherical anisotropic PML media [10], [11] provides only an *approximately* matched layer.

It should be noted that, for the cylindrical case, an alternative derivation of anisotropic PML was carried out on [4] through a graphical construction. The constitutive tensors obtained there coincide with those of our analysis.

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## II. From Complex Space to Anisotropic PML

In [2] and [3] it was shown how the analytic continuation of the frequency-domain Maxwell's equations to complex space achieves the reflectionless absorption of the electromagnetic waves. This motivated the development of PML-FDTD algorithms in cylindrical and spherical grids [3]. In analogy to the Cartesian PML case [6], [7], the objective here is to derive a mapping of the non-Maxwellian fields of [3] to a set of Maxwellian fields on cylindrical and spherical anisotropic PML media and to determine the constitutive parameters of such media.

To introduce this general approach on a simpler setting and for completeness, we first discuss it briefly for Cartesian coordinates.

The $x$-component of the Faraday equation on complex space [2] reads as ($e^{-i\omega t}$ convention)

$$i\omega\mu H_x^c = \frac{\partial E_z^c}{\partial \tilde{y}} - \frac{\partial E_y^c}{\partial \tilde{z}} = \frac{1}{s_y}\frac{\partial E_z^c}{\partial y} - \frac{1}{s_z}\frac{\partial E_y^c}{\partial z} \tag{1}$$

where $s_\zeta$ ($\zeta = x, y, z$) are the stretching variables [2], [12]. The fields in (1) do not satisfy Maxwell's equations when $s_\zeta \neq 1$ (i.e., inside the PML), and to make this fact more explicit the superscript $c$ is added onto the field variables. However, if we multiply (1) by $s_y s_z$ and using the fact that $s_\zeta$ and $\partial/\partial\zeta'$ commute when $\zeta \neq \zeta'$, we arrive at

$$i\omega\mu \frac{s_y s_z}{s_x}(s_x H_x^c) = \frac{\partial}{\partial y}(s_z E_z^c) - \frac{\partial}{\partial z}(s_y E_y^c). \tag{2}$$

If we then repeat the same procedure for the other components of the curl equations and introduce a new set of fields $E_\zeta^a = s_\zeta E_\zeta^c$ and $H_\zeta^a = s_\zeta H_\zeta^c$, then this new set of fields obeys Maxwell's equations on an anisotropic medium of constitutive parameters $\bar{\bar{\mu}} = \mu\bar{\bar{\Lambda}}$ and $\bar{\bar{\epsilon}} = \epsilon\bar{\bar{\Lambda}}$, with

$$\bar{\bar{\Lambda}} = \hat{x}\hat{x}\left(\frac{s_y s_z}{s_x}\right) + \hat{y}\hat{y}\left(\frac{s_z s_x}{s_y}\right) + \hat{z}\hat{z}\left(\frac{s_x s_y}{s_z}\right) \tag{3}$$

as obtained in [7]. This is the most general form for the constitutive tensors on the Cartesian anisotropic PML formulation (corresponding to the PML medium at corner interfaces [7]). In a single planar interface case, only the stretching coordinate normal to the interface has $s_\zeta \neq 1$ and the medium is uniaxial, as first derived in [6].

In the cylindrical system, we write the Faraday's equation on complex space as

$$i\omega\mu H_\rho^c = \frac{1}{\tilde\rho}\frac{\partial E_z^c}{\partial\phi} - \frac{\partial E_\phi^c}{\partial\tilde z} \tag{4a}$$

$$i\omega\mu H_\phi^c = \frac{\partial E_\rho^c}{\partial\tilde z} - \frac{\partial E_z^c}{\partial\tilde\rho} \tag{4b}$$

$$i\omega\mu H_z^c = \frac{1}{\tilde\rho}\frac{\partial}{\partial\tilde\rho}(\tilde\rho E_\phi^c) - \frac{1}{\tilde\rho}\frac{\partial E_\rho^c}{\partial\phi}. \tag{4c}$$

$\tilde z$ and $\tilde\rho$ are complex spatial variables defined as

$$\tilde z = z_0 + \int_{z_0}^{z} s_z(z')dz',\ \tilde\rho = \rho_0 + \int_{\rho_0}^{\rho} s_\rho(\rho')d\rho' \tag{5}$$

so as to achieve the reflectionless absorption of the electromagnetic waves in the $z$- and $\rho$-directions, respectively [2], [3]. From (5) we have that $\partial/\partial\tilde\rho = (1/s_\rho)\partial/\partial\rho$ and $\partial/\partial\tilde z = (1/s_z)\partial/\partial z$. If we substitute these last identities on (4a)–(4c), multiply (4a) by $s_z(\tilde\rho/\rho)$, (4b) by $s_z s_\rho$, and (4c) by $s_\rho(\tilde\rho/\rho)$, then (4) can be recast into the following form:

$$i\omega\mu\left[\left(\frac{\tilde\rho}{\rho}\right)\frac{s_z}{s_\rho}\right](s_\rho H_\rho^c) = \frac{1}{\rho}\frac{\partial}{\partial\phi}(s_z E_z^c) - \frac{\partial}{\partial z}\left(\frac{\tilde\rho E_\phi^c}{\rho}\right) \tag{6a}$$

$$i\omega\mu\left[\left(\frac{\rho}{\tilde\rho}\right)s_z s_\rho\right]\left(\frac{\tilde\rho H_\phi^c}{\rho}\right) = \frac{\partial}{\partial z}(s_\rho E_\rho^c) - \frac{\partial}{\partial\rho}(s_z E_z^c) \tag{6b}$$

$$i\omega\mu\left[\left(\frac{\tilde\rho}{\rho}\right)\frac{s_\rho}{s_z}\right](s_z H_z^c) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(\frac{\tilde\rho E_\phi^c}{\rho}\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial\phi}(s_\rho E_\rho^c). \tag{6c}$$

From (6a)–(6c) and their duals (Ampere's equation), we see that a new set of fields defined by $E_\rho^a = s_\rho E_\rho^c$, $E_\phi^a = (\tilde\rho/\rho)E_\phi^c$, $E_z^a = s_z E_z^c$ (similarly for the $\vec H$ field), obeys Maxwell's equations on an anisotropic medium of constitutive parameters $\bar{\bar\mu} = \mu\bar{\bar\Lambda}$ and $\bar{\bar\epsilon} = \epsilon\bar{\bar\Lambda}$, with

$$\bar{\bar\Lambda} = \hat\rho\hat\rho\left(\frac{\tilde\rho}{\rho}\right)\left(\frac{s_z}{s_\rho}\right) + \hat\phi\hat\phi\left(\frac{\rho}{\tilde\rho}\right)(s_z s_\rho) + \hat z\hat z\left(\frac{\tilde\rho}{\rho}\right)\left(\frac{s_\rho}{s_z}\right). \tag{7}$$

If we set $s_z = 1$ everywhere, this tensor is identical to the one derived in [4] through a graphical approach. By *defining* $s_\phi = (\tilde\rho/\rho)$, then the above tensor and the field mapping equations have the same *formal* appearance as in the Cartesian case. Both Maxwellian and non-Maxwellian formulations satisfy the same boundary conditions on the continuity of tangential fields across the PML interface. This is because the corresponding tangential fields differ by factors that are continuous across PML interfaces $z = z_0$ and $\rho = \rho_0$.[1] Because of this, the perfect matching condition for one of the formulations follows automatically from the other, a requirement of consistency. This perfect matching condition was independently demonstrated for the complex-space formulation in [2] and [3], and for the anisotropic-medium formulation in [4].

However, the normal components satisfy *in general* different boundary conditions, since $s_z$ and $s_\rho$ are not necessarily continuous (although in the practical numerical implementation, this is usually imposed to minimize spurious reflections due to discretization). Moreover, the $(E_\zeta^c, H_\zeta^c,)$ fields do not satisfy divergence-free conditions inside the PML. Also, it is evident that if $\rho \to \infty$, then $(\tilde\rho/\rho) \to 1$, and we recover the Cartesian case. From this analysis, the reason is clear why the constitutive tensors introduced in [10] with $s_z = 1$ do not correspond to a true PML and approximates a PML only in the limit of large radius: because the radial scaling factor $(\tilde\rho/\rho)$ is not included.

In the spherical system, we write the Faraday's equation on complex space as

$$i\omega\mu H_r^c = \frac{1}{\tilde r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi^c) - \frac{\partial E_\theta^c}{\partial\phi}\right] \tag{8a}$$

$$i\omega\mu H_\theta^c = \frac{1}{\tilde r\sin\theta}\frac{\partial E_r^c}{\partial\phi} - \frac{1}{\tilde r}\frac{\partial}{\partial\tilde r}(\tilde r E_\phi^c) \tag{8b}$$

$$i\omega\mu H_\phi^c = \frac{1}{\tilde r}\frac{\partial}{\partial\tilde r}(\tilde r E_\theta^c) - \frac{1}{\tilde r}\frac{\partial E_r^c}{\partial\theta} \tag{8c}$$

with $\tilde r$ being defined analogously as in (5) [3]. If we substitute $\partial/\partial\tilde r = (1/s_r)\partial/\partial r$ on (8b)–(8c), multiply (8a) by $(\tilde r/r)^2$, and (8b)–(8c) by $s_r(\tilde r/r)$, then (8) can be recast as

$$i\omega\mu\left[\left(\frac{\tilde r}{r}\right)^2\frac{1}{s_r}\right](s_r H_r^c) = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\tilde r E_\phi^c}{r}\right)\right] - \frac{\partial}{\partial\phi}\left(\frac{\tilde r E_\theta^c}{r}\right)\right\} \tag{9a}$$

$$i\omega\mu s_r\left(\frac{\tilde r H_\theta^c}{r}\right) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(s_r E_r^c) - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\tilde r E_\phi^c}{r}\right)\right] \tag{9b}$$

$$i\omega\mu s_r\left(\frac{\tilde r H_\phi^c}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\tilde r E_\theta^c}{r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial\theta}(s_r E_r^c) \tag{9c}$$

and similarly for the Ampere's equation. A set of fields defined by $E_r^a = s_r E_r^c$, $E_\phi^a = (\tilde r/r)E_\phi^c$ $E_\theta^a = (\tilde r/r)E_\theta^c$ (similarly for the $\vec H$ field) obeys Maxwell's equations on an anisotropic medium of constitutive parameters $\bar{\bar\mu} = \mu\bar{\bar\Lambda}$ and $\bar{\bar\epsilon} = \epsilon\bar{\bar\Lambda}$, with

$$\bar{\bar\Lambda} = \hat r\hat r\left(\frac{\tilde r}{r}\right)^2\left(\frac{1}{s_r}\right) + \left(\hat\theta\hat\theta + \hat\phi\hat\phi\right)s_r. \tag{10}$$

By *defining* $s_\theta = s_\phi = (\tilde r/r)$, then this tensor and the field mapping equations have the same formal appearance as the previous ones. The continuity of $\tilde r/r$ everywhere implies that both formulations will satisfy the same boundary conditions on the continuity of the tangential fields across a PML interface at $r = r_0$. The perfectly matched condition for one of the formulations then follows automatically from the other. For the spherical case, the perfectly matched condition was demonstrated for the complex-space formulation in [2] and [3], and can be independently demonstrated for the anisotropic formulation along very similar lines to the cylindrical case as done in [4]. The same observations made on the cylindrical

[1] Since $s_\rho$ is a function of $\rho$ only, $s_z$ is a function of $z$ only, and $\tilde\rho/\rho$ is continuous everywhere as implied by (5).

case about normal boundary conditions and the limit of infinite radius also apply here.

It can also be easily verified that $\tilde{\nabla}\cdot\vec{E}^c = 0$ implies $\nabla\cdot\left(\bar{\bar{\Lambda}}\cdot\vec{E}^a\right) = 0$ in all three coordinate systems.

## III. Final Discussion

A simple and systematic way to derive anisotropic PML's from the complex space Maxwell's Equations in different coordinate systems has been presented. Through this analysis, the constitutive tensors of the PML media and the mapping equations relating the fields of the complex space and anisotropic formulations are derived. From the mapping equations, it was shown that the fields in the complex space and anisotropic formulations differ by factors that are equal to unity inside the physical region and are continuous across PML interfaces for the tangential field components, implying an equivalence of perfectly matched conditions.

From the mapping equations and by using the usual form of frequency dependence for the stretching coordinates $s_\zeta = a_\zeta + i\sigma_\zeta/\omega$ (with $a_\zeta \geq 1, \sigma_\zeta \geq 0$, and $\zeta = x, y, z, \rho, r$) , the time-domain relationship between the fields inside the PML can be written in a generic form as

$$\frac{\partial}{\partial t}E_\zeta^a = \left(\alpha\frac{\partial}{\partial t} + \beta\right)E_\zeta^c \tag{11}$$

(with $\zeta = x, y, z, \rho, \phi, r, \theta$) and similarly for the H fields. The parameters $\alpha$ and $\beta$ are, in any case, positive, except for the parameter $\beta$ in the case of the *angular* components on a *convex* PML surface (where $\tilde{\rho}$ and $\tilde{r}$ have negative imaginary parts and $\bar{\bar{\Lambda}}$ has possibly poles on the upper half of the complex space). This fact implies an inherent asymmetry between the convex/concave cylindrical and spherical PML's (defined over inner/outer domains). This, together with the causality constraints to be observed by $\bar{\bar{\Lambda}}$ [11], [13], could have severe consequences on the stability of the FDTD scheme. Although simulations of cases using outer domain (concave surface) cylindrical and spherical PML's [3] were found to be stable, it is still an open question for the convex surface PML.

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